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S-ASYMPTOTICALLY ω-PERIODIC MILD SOLUTIONS FOR THE SYSTEMS OF DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT IN BANACH SPACES

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Abstract. By using of the Banach fixed point theorem, the theory of a strongly continuous semigroup of operators and resolvent operator, we investigate the existence and uniqueness of S-asymptotically ω-periodic mild solutions for some differential (integrodifferential) equations with piecewise constant argument when specially ω is an integer.

1. Introduction

The theory of almost periodic functions was introduced in the literature around 1924-1926 with the pioneering work the Denish mathematician Harald Bohr. Many authors have furthermore generalized in different directions the notion of almost periodicity for more realistic decription to real world phenomenon around us. Differential equations with piecewise constant argument arise in an attempt to the theory of functional differential equations with continuous argument to differential equations with discontinuous arguments. The strong interest in these equations is the fact that they describe hybrid dynamic systems (a continuous and discrete combination) and, therefore, combine properties of both differential and difference equations. Furthermore these equations have the structure of continuous dynamical systems in intervals of unit length. These equations are thus similar in structure to those found in

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certain sequential-continuous models of disease dynamics as treated by Busenberg and Cooke since 1982 [2]. The first contribution is due to Cooke and Wiener in 1984 [4] and Shah and Wiener in 1983 [16]. In [5], Piao Daxiong investigated the existence of pseudo almost periodic solutions to the system of differential equations with piecewise constant argument (EPCA) of the form

$$u'(t) = Au(t) + Bu([t]) + f(t),\, t \in R,$$

where A, B are constant matrices and A is nonsingular. In [17], Nguyen Van Minh and Tran Tat Dat gave suffcient spectral conditions for the almost automorphy of bounded solutions to differential equations with piecewise constant argument of the form

$$u'(t) = Au([t]) + f(t),\, t \in R,$$

where A is a bounded linear operator in X and f in a X-valued almost automorphic function.

Let us give a general description of the systems with piecewise con stant argument [1]. Since the main peculiarity is the involvement of piecewise constant functions as an arguments, it is reasonable to give at first the description on these functions.

Fix an interval $J \in \mathbb{R}$. Denote by $\theta = \{\theta_i\}$, $\theta \in J$, a strictly ordered sequence of real numbers such that the set of indices i is an interval of $\mathbb{Z}$. Let also $\zeta = \{\zeta_i\}$ be another sequence of elements of J. We do not impose any restriction on ζ.

We say that a function, which is defined on J, is of the η-type, and denote it η(t) if it is equal to ζ_i if $\theta_i \leq t \leq \theta_{i+1}$. This is the most general type of argument functions.

Specifically, we shall define the following η-type functions. We say that a function is of the β-type, and denote it $\beta(t)$ if $\zeta_i = \theta_i$.

For example, the greatest integer function $[t]$ is a $\beta(t)$ function with $\theta_i = i$, $i \in \mathbb{Z}$.

In this paper, specifically we consider that $\beta(t) = [t]$. Recently Dimbour and Mado [9] worked that the existence of S-asymptotically ω-periodic solutions for the following equation

$$u'(t) = Au(t) + A_0u([t]) + f(t, u(t)),\ u(0) = u_0,$$

and Dimbour and Valmorin [10] worked that the existence and uniquenss of asymptotically antiperiodic solutions in a Banach space for the following equation

$$u'(t) = Au(t) + A_0u([t]) + f(t, u([t]),\ u(0) = u_0,$$

where ω is an integer.

Stimulated by above work, we investigate that the existence and uniqueness of S-asymptotically ω-periodic solutions for the following system

$$u'(t) = Au(t) + A_0u([t]) + g(t, u([t])),\ u(0) = u_0, \tag{1.1}$$

where ω is an integer, A_0 is a bounded linear operator and A is the infinitesimal generator of an exponentially stable C_0-semigroup acting on X, $[\cdot]$ is the largest integer function and g is an appropriate function that will be given later.

2. Preliminaries

To work S-asymptotically ω-periodic functions, it is very convenient to introduce the following notations: For a Banach space X,

$$\begin{aligned}
C(\mathbb{R}_+, X) &= \{x : \mathbb{R}_+ \to X : x \text{ is continuous } \}\\
C_b(\mathbb{R}_+, X) &= \{x \in C(\mathbb{R}_+, X) : \sup_{t\geq 0} \|x(t)\| < \infty\}\\
C_0(\mathbb{R}_+, X) &= \{x \in C_b(\mathbb{R}_+, X) : \lim_{t\to\infty} \|x(t)\| = 0\}\\
C_\omega(\mathbb{R}_+, X) &= \{x \in C_b(\mathbb{R}_+, X) : x \text{ is } \omega\text{- periodic } \}
\end{aligned}$$

endowed with the norm $\|f\|_\infty ; = \sup_{t\geq 0} \|f(t)\|$.

We introduce some definitions and lemmas well known from our references [2], [11] and [12].

Definition 2.1. A function $f \in C_b(\mathbb{R}_+, X)$ is called *almost periodic* if for every $\epsilon > 0$, if there exists an l such that every interval of length $l(\epsilon)$ contains a number τ with property that

$$\|f(t + \tau) - f(t)\| < \epsilon \ \text{ for every } \ t \in \mathbb{R}.$$

Many authors have furthermore generalized the notion of almost periodicity in different directions.

Definition 2.2. A function $f \in C_b(\mathbb{R}_+, X)$ is called *asymptotically almost periodic* if there exist $g \in AP(\mathbb{R}, X)$ and $\phi \in C_0(\mathbb{R}_+, X)$ such that $f = g + \phi$. Also f is said to be *asymptotically ω-periodic* when $g \in C_\omega(\mathbb{R}_+, X)$.

Definition 2.3. A function $f \in C_b(\mathbb{R}_+, X)$ is said to be *S-asymptotically ω-periodic* if there exists an $\omega > 0$ such that

$$\lim_{t\to\infty} \|f(t + \omega) - f(t)\| = 0.$$

In this case, we say that ω is an asymptotic period of f and f is S-asymptotically ω-periodic. Denote by $SAP_\omega(X)$ the set of such functions. It is clear that $(SAP_\omega(X), \|\cdot\|_\infty)$ is a Banach space (see [11]).

Let W be a Banach space.

Definition 2.4. A function $f \in C(\mathbb{R}_+ \times W, X)$ is called *uniformly S-asymptotically ω-periodic* on bounded sets if for every bounded set $K \subset W$, the set $\{f(t,x) : t \geq 0, x \in K\}$ is bounded and

$$\lim_{t\to\infty}(f(t+\omega,x) - f(t,x)) = 0$$

uniformly for $x \in K$.

Definition 2.5. A function $f \in C(\mathbb{R}_+ \times W, X)$ is called *asymptotically uniformly continuous* on bounded sets if for every $\epsilon > 0$ and all bounded set $K \subset W$ there exist constants $L_{K,\epsilon} \geq 0$ and $\delta_{K,\epsilon} > 0$ such that

$$\|f(t,x) - f(t,y)\|_X \leq \epsilon, \ t \geq L_{K,\epsilon},$$

when $\|x-y\|_W \leq \delta_{K,\epsilon}, \ x, y \in K$.

Definition 2.6. Let $T(t)$ be the C_0-semigroup generated by A and $g \in L^1(\mathbb{R}_+, X)$. The function $u(t) \in C(\mathbb{R}_+, X)$ given by

$$u(t) = \quad T(t)u_0 + \int_0^t T(t-s)A_0u([s])ds + \int_0^t T(t-s)g(s,u([s]))ds,$$

is the mild solution of $Eq.(1.1)$.

A composition theorem is a great interest when it comes to dealing with differential equations with piecewise constant arguments.

Theorem 2.7. [12] *Assume that $f \in C(\mathbb{R}_+ \times W, Z)$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. If $u \in SAP_\omega(W)$, then the function $t \to f(t,u(t))$ belong to $SAP_\omega(Z)$.*

3. Existence results for S-asymptotically ω-periodic mild solution

To establish our results, we introduce the following conditions.

(H_1) There are positive constants M, μ such that

$$\|T(t)\| \leq Me^{-\mu t} \text{ for all } t \geq 0.$$

(H_2) Assume that $g : \mathbb{R}_+ \times X \to X$ is uniformly S-asymptotically ω-periodic on bounded sets that verifies the Lipschitz condition

$$\|g(t,x) - g(t,y)\| \le L\|x-y\| \quad \text{for all } x, y \in X,\ t \ge 0.$$

DEFINITION 3.1. A solution of $Eq.(1.1)$ on $\mathbb{R}_+$ is a function $x(t)$ that satisfies the conditions:

1. $x(t)$ is continuous on $\mathbb{R}_+$.
2. The derivative $x^{'}(t)$ exists at each point $t \in \mathbb{R}_+$, with possible exception of the point $t \in \mathbb{R}_+$ where onesided derivatives exists.
3. $Eq.(1.1)$ is satisfied on each interval $[n, n+1)$ with $n \in \mathbb{N}$.

LEMMA 3.2. [9] *We assume that the hypothesis (H_1) is satisfied. Then the function $L(t)$ defined by*

$$L(t) = T(t)u_0,$$

where the function $L(t)$ is locally integrable on $\mathbb{R}_+$ belongs to $SAP_\omega(X)$.

LEMMA 3.3. *Assume that the hypothese (H_1) is satisfied and $\omega \in \mathbb{N}$. Define the nonlinear operator Γ as follows, for each $\phi \in SAP_\omega(X)$*

$$(\Gamma_1\phi)(t) = \int_0^t T(t-s)A_0\phi([s])ds.$$

Then the operator Γ_1 maps $SAP_\omega(X)$ into itself.

LEMMA 3.4. *Assume that the hypothesis (H_2) is satisfied and $\omega \in N$. Define the nonlinear operator Γ as follows, for each $\phi \in SAP_\omega(X)$*

$$(\Gamma\phi)(t) = \int_0^t T(t-s)g(s, \phi([s]))ds.$$

Then the operator Γ maps $SAP_\omega(X)$ into itself.

Proof. Let $v = \int_0^t T(t-s)g(s,\phi([s]))ds$. Then

$$\begin{aligned}
&v(t+\omega) - v(t) \\
&= \int_0^{t+\omega} T(t+\omega-s)g(s,\phi([s]))ds - \int_0^t T(t-s)g(s,\phi([s]))ds \\
&= \int_0^{\omega} T(t+\omega-s)g(s,\phi([s]))ds + \int_\omega^{t+\omega} T(t+\omega-s)g(s,\phi([s]))ds \\
&\quad - \int_0^t T(t-s)g(s,\phi([s]))ds \\
&= \int_t^{t+\omega} T(s)g(t+\omega-s,\phi([t+\omega-s]))ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t T(t-s)[g(s+\omega,\phi[s+\omega])-g(s,\phi(s)]ds\\
&=\int_t^{t+\omega} T(s)g(t+\omega-s,\phi([t+\omega-s]))ds\\
&\quad+\int_0^t T(s)[g(t+\omega-s,\phi([t+\omega-s]))-g(t-s,\phi([t-s])]ds\\
&=\int_0^T T(s)[g(t+\omega-s,\phi([t+\omega-s]))-g(t-s,\phi([t+\omega-s]))]ds\\
&\quad+\int_0^T T(s)[g(t-s,\phi([t+\omega-s])-g(t-s,\phi([t-s]))]ds\\
&\quad+\int_T^t T(s)[g(t+\omega-s,\phi([t+\omega-s])-g(t-s,\phi([t-s]))]ds\\
&\quad+\int_t^{t+\omega} T(s)g(t+\omega-s,\phi([t+\omega-s]))ds\\
&=I_1+I_2+I_3+I_4.
\end{aligned}
$$

Since the function $u(.)$ is bounded, it follows from the Definition 2.4 that $C=\sup_{s\geq 0}\|g(u,u(s))\|<\infty$.

Put $\theta := M\sup_{t\geq 0}\int_0^t e^{-\mu(t-s)}L(s)ds$.

Also, since $\phi\in SAP_\omega(X)$, for each $\epsilon>0$, there exists $T^{'}(>t)$ such that $\|\phi(t+\omega)-\phi(t)\|<\epsilon$. Put $T=[T^{'}]+1$, for $\epsilon>0$, let $\epsilon'=min\{\frac{\mu}{M},\frac{1}{\theta}\}(\frac{\epsilon}{3})$.

Choose $T>0$ such that the following conditions hold:

(i) $e^{-\mu T}\leq \epsilon\mu/9CM$.
(ii) $\|\phi([t+\omega])-\phi([t])\|\leq\epsilon'$,
(iii) $\|g(t+\omega,x)-g(t,x)\|\leq\epsilon'$,

for all $t\geq T$.

Let $t\geq 2T$. Since $t-s\geq t-T\geq T$ for $0\leq s\leq T$, we get

$$
\begin{aligned}
\|I_1\|&=\Big\|\int_0^T T(s)[g(t+\omega-s,\phi([t-s]+\omega))\\
&\qquad\qquad -g(t-s,\phi([t-s]+\omega)]ds\Big\|\\
&\leq M\epsilon'\int_0^T e^{-\mu s}ds\leq \epsilon'\cdot\frac{M}{\mu}=\frac{\mu}{M}\cdot\frac{\epsilon}{3}\cdot\frac{M}{\mu}=\frac{\epsilon}{3},
\end{aligned}
$$

$$
\begin{aligned}
\|I_2\| &= \Big\| \int_0^T T(s)[g(t-s,\phi([t-s]+\omega)-g(t-s,\phi([t-s]))ds\Big\| \\
&\le \epsilon' M\int_0^T e^{-\mu s}L(t-s)ds \le \epsilon' M\int_0^t e^{-\mu s}L(t-s)ds \\
&\le \epsilon'\theta = (\frac{1}{\theta}\frac{\epsilon}{3})\theta = \frac{\epsilon}{3}, \\
\|I_3\| &= \Big\| \int_T^t T(s)[g(t+\omega-s,\phi(t-s]+\omega)-g(t-s,\phi([t-s])]ds\Big\| \\
&\le M\int_T^t e^{-\mu s}2Cds \le \frac{2CM}{\mu}e^{-\mu T}, \\
\|I_4\| &= \Big\| \int_t^{t+\omega} T(s)g(t+\omega-s,\phi([t-s]+\omega)ds\Big\| \\
&\le M\int_t^{t+\omega} e^{-\mu s}Cds \le \frac{CM}{\mu}e^{-\mu T}, \\
\||I_3\|+\|I_4\| &\le \frac{2CM}{\mu}e^{-\mu T}+\frac{CM}{\mu}e^{-\mu T} \le \frac{3CM}{\mu}e^{-\mu T} \\
&\le \frac{3CM}{\mu}\cdot\frac{\epsilon\mu}{9CM} = \frac{\epsilon}{3}.
\end{aligned}
$$

Combining these estimates,

$$\|v(t+\omega)-v(t)\| \le \epsilon$$

for $t \ge 2T$. Therefore we know $v \in SAP_\omega(X)$. □

THEOREM 3.5. *Assume that the hypotheses (H_1) and (H_2) are satisfied and ω is an integer. If $\frac{||A_0||M+LM}{\mu} < 1$, then the Eq.(1.1) has a unique S-asymptotically ω-periodic mild solution.*

Proof. Define the operator $\Gamma : SAP_\omega(X) \to SAP_\omega(X)$ by

$$\Gamma u(t) = T(t)u_0 + \int_0^t T(t-s)A_0u([s])ds + \int_0^t T(t-s)g(s,u[s]))ds$$

for $t \ge 0$. Then

$$v(t) = \int_0^t T(t-s)g(s,u[s])ds$$

belongs to $SAP_\omega(X)$ by Lemma 3.3. The Lemma 3.2 shows that $T(t)u_0$ is an S-asymptotically ω-periodic.

Then Γ maps $SAP_\omega(X)$ into itself. Γ is well defined. For $x, y \in SAP_\omega(X)$, we have

$$\begin{aligned}
\|\Gamma x(t) - \Gamma y(t)\| &\leq \left\| \int_0^t T(t-s)A_0(x([s]) - y([s]))ds \right\| \\
&+ \left\| \int_0^t T(t-s)(g(s, x([s])) - g(s, y([s])))ds \right\| \\
&\leq \int_0^t Me^{-\mu}\|A_0\|ds\|x - y\|_\infty \\
&+ \int_0^t LMe^{-\mu}ds\|x - y\|_\infty \\
&\leq \frac{\|A_0\|M + LM}{\mu}\|x - y\|_\infty
\end{aligned}$$

Therefore Γ is a contraction and there exists a unique fixed point $u \in SAP_\omega(X)$. This function u is an S-asymptotically ω-periodic mild solution of $Eq.(1.1)$. □

4. Existence results of S-asymptotically ω-periodic mild solution for some partial integrodifferential equation

Consider the following partial integrodifferential equation with piecewise constant argument

$$\begin{aligned}
u'(t) &= Au(t) + \int_0^t B(t-s)u(s)ds + f(t, u([t])),\ t \geq 0, \\
u(0) &= u_0, \qquad (4.1)
\end{aligned}$$

where $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0 -semigroup on a Banach space X and $B(t) : D(B(t) \subseteq X \to X$ for $t \geq 0$ are densely defined closed linear operators in a Banach space $(X, \|\cdot\|)$.

We assume that $D(A) \subset D(B(t))$ for every $t \geq 0$ and that $f : [0, \infty) \times X \to$ is a suitable function.

To obtain our results, we use the theory of resolvent operators. This theory is related to abstract integrodifferential equations in a similar manner as the semigroup theory is related to first order linear abstract partial differential equations.

$[D(A)]$ represents the space $D(A)$ endowed with graph norm given by $\|x\|_A = \|x\| + \|Ax\|$.

Definition 4.1. [6] A family $\{R(t) : t \geq 0\}$ of continuous linear operators on X is called a resolvent operator for $Eq.(4.1)$ if the following conditions are fulfilled.

1. For each $x \in X$, $R(0)x = x$ and $R(\cdot)x \in C([0,\infty); X)$.
2. The map $R : [0,\infty) \to L([D(A)])$ is strongly continuous .
3. For each $y \in D(A)$, the function $t \to R(t)y$ is continuously differentiable and

$$\begin{aligned}\frac{d}{dt}R(t)y &= AR(t)y + \int_0^t B(t-s)R(s)yds \\ &= R(t)Ay + \int_0^t R(t-s)B(s)yds, \quad t \geq 0.\end{aligned}$$

We assume that there exists a unique resolvent operator for $Eq.(4.1)$. Motivated from work of [3], we define the following definition of mild solution.

(H_3) There are positive constants M, μ such that

$$\|R(t)\| \leq Me^{-\mu t} \text{ for all } t \geq 0.$$

Definition 4.2. A function $u \in C(\mathbb{R}_+, X)$ is called a *mild solution* of $Eq.(4.1)$ if u satisfies

$$\begin{aligned}u(t) &= R(t)u_0 + \int_0^t R(t-s)f(s, u([s]))ds, \; t \geq 0, \\ u(0) &= u_0 \in X.\end{aligned}$$

Lemma 4.3. [12] *Assume that $f \in C(\mathbb{R}_+ \times W; X)$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. If $u \in SAP_\omega(W)$, then the function $t \to f(t, u(t))$ belongs to $SAP_\omega(X)$.*

Lemma 4.4. *Let $u \in SAP_\omega(X)$. Then*

$$v(t) = \int_0^t R(t-s)u([s])ds \in SAP_\omega(X).$$

Proof.

$$\begin{aligned}\|v(t)\| &\leq \int_0^t \|R(t-s)\|\|u([s])\|ds \\ &\leq \int_0^t Me^{-\mu(t-s)}\|u([s])\|ds \\ &\leq \frac{M}{\mu}\|u\|_\infty.\end{aligned}$$

Thus $\|v\|_\infty = sup_{t\geq 0}\|v(t)\| \leq \frac{M}{\mu}\|u\|_\infty$, where $v \in C_b(R_+;X)$.

Since $u \in SAP_\omega(X)$, for $\epsilon > 0$, we select $T' > 0$ such that

$$\|u(t+\omega) - u(t)\| \leq \epsilon \text{ for all } t \geq T'$$

and

$$\int_{T'}^{\infty} e^{-\mu s} ds \leq \epsilon.$$

Put $T = [T']+1$ and let $t \geq 2T$. Since $t-s \geq t-T \geq T$ for $0 \geq s \geq T$, we get:

$$\begin{aligned}
&v(t+\omega) - v(t) \\
&\quad = \int_0^{t+\omega} R(t+\omega-s)u([s])ds - \int_0^t R(t-s)u([s])ds \\
&\quad = \int_0^{\omega} R(t+\omega-s)u([s])ds + \int_{\omega}^{t+\omega} R(t+\omega-s)u([s])ds \\
&\qquad - \int_0^t R(t-s)u([s])ds \\
&\quad = -\int_{t+\omega}^{t} R(s)u([t+\omega-s])ds + \int_0^t R(t-s)u([s+\omega])ds \\
&\qquad - \int_0^t R(t-s)u([s])ds \\
&\quad = \int_t^{t+\omega} R(s)u([t+\omega-s])ds + \int_0^t R(t-s)[u([s+\omega]) - u([s])]ds \\
&\quad = \int_t^{t+\omega} R(s)u([t+\omega-s])ds + \int_0^T R(t-s)[u([s+\omega]) - u([s])]ds \\
&\qquad + \int_T^t R(t-s)[u([s+\omega]) - u([s])]ds.
\end{aligned}$$

Hence, for $t \geq 2T$ we obtain

$$\begin{aligned}
&\|v(t+\omega) - v(t)\| \\
&\leq \int_0^{t+\omega} \|R(s)\|\|u([t+\omega-s])\|ds + \int_0^T \|R(t-s)\|[u([s+\omega]) - u([s])]\|ds \\
&\quad + \int_T^t \|R(t-s)\|[u([s+\omega]) - u([s])\|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{t+\omega} \|R(s)\|\|u([t+\omega-s])\|ds \\
&\quad + \int_{t-T}^{t} \|R(t-s)\|[u([s+\omega]) - u([s])\|ds \\
&\quad - \int_0^{t-T} \|R(s)\|\|[u([t-s+\omega]) - u([t-s])]\|ds \\
&\leq \int_0^{t+\omega} \|R(s)\|\|u([t+\omega-s])\|ds \\
&\quad + \int_{t-T}^{t} \|R(s)\|\|(u([t-s]+\omega) - u([t-s]))\|ds \\
&\quad + \int_0^{t-T} \|R(s)\|\|(u([t-s]+\omega) - u([t-s]))\|ds \\
&\leq M\|u\|_\infty \int_t^{t+\omega} e^{-\mu s}ds + 2M\|u\|_\infty \int_{t-T}^{t} e^{-\mu s}ds + \epsilon M \int_0^{t-T} e^{-\mu s}ds \\
&\leq M\|u\|_\infty \int_t^{t+\omega} e^{-\mu s}ds + 2M\|u\|_\infty \int_T^{t} e^{-\mu s}ds + \epsilon M \int_0^{t} e^{-\mu s}ds \\
&\leq 3M\|u\|_\infty \int_T^{\infty} e^{-\mu s}ds + \frac{M}{\mu}\epsilon \\
&= M(3\|u\|_\infty + \frac{1}{\mu})\epsilon.
\end{aligned}
$$

Hence $\lim_{t\to\infty}(v(t+\omega) - v(t)) = 0$, which completes the proof. □

THEOREM 4.5. *Assume that $f : \mathbb{R}_+ \times X \to X$ uniformly S-asymptotically ω-periodic on bounded sets function that verifies the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|,$$

for all $x, y \in X$ and every $t \geq 0$.

If $LM/\mu < 1$, then Eq.(4.1) has a unique S-asymptotically ω-periodic mild solution.

Proof. Define the operator $\Gamma : SAP_\omega(X) \to SAP_\omega(X)$ given by

$$\Gamma u(t) = R(t)x_0 + \int_0^t R(t-s)f(s,u[s]))s,\ t \geq 0.$$

By Lemma 4.3, $f(\cdot, u(\cdot)) \in SAP_\omega(X)$. Let $v(t) = \int_0^t R(t-s)f(s, u[s])ds$. Then

$$\begin{aligned} \|v(t)\| &\le \int_0^t \|R(t-s)\|\|f(s, u([s]))\|ds \\ &\le \int_0^t Me^{-\mu(t-s)}\|f(s, u([s]))\|ds \\ &\le \frac{M}{\mu}\int_0^t \|f(s, u([s]))\|ds. \end{aligned}$$

Thus $\|v\|_\infty = sup_{t\ge 0}\|v(t)\| \le \frac{M}{\mu}\|u\|_\infty$. Hence $v \in C_b(\mathbb{R}_+; X)$.

Note that for any $\epsilon > 0$, we select $T > 0$ such that

$$\|u(t+\omega) - u(t)\| \le \epsilon \text{ for all } t \ge T$$

and $\int_T^\infty e^{-\mu s}ds \le \epsilon$.

For any $t \ge 2T$, by the similar calculation of the proof in Lemma 4.4.

$$\|v(t+\omega) - v(t)\| \le M(3\|g\|_\infty + \frac{LM}{\mu}\|u\|_\infty)\epsilon.$$

Thus $v(t) \in SAP_\omega(X)$. Also, since $R(\cdot)x_0 \in SAP_\omega(X)$ (see in Lemma 3.2), $\Gamma u(t) \in SAP_\omega(X)$ (Γ is well defined!). We will show that Γ is a contraction.

For $u, v \in SAP_\omega(X)$, we have

$$\begin{aligned} \|\Gamma u(t) - \Gamma v(t)| &\le \|\int_0^t R(t-s)(f(s, u([s]) - f(s, v([s]))ds\| \\ &\le \int_0^t \|R(t-s)\|\|f(s, u([s]) - f(s, v([s]))\|ds \\ &\le \int_0^t LMe^{-\mu(t-s)}\|u([s]) - v([s])\|ds \\ &\le \frac{LM}{\mu}\|u - v\|_\infty. \end{aligned}$$

Therefore Γ is a contraction. By the Banach fixed point theorem, there exists a unique fixed point $u \in SAP_\omega(X)$. This function u is an S-asymptotically ω-periodic mild solution of $Eq.(4.1)$. The proof is complete. □

5. Application

As an application [10], we give an example as follows.

$$\frac{\partial u}{\partial t}(t,x) = \begin{cases} \frac{\partial^2 u}{\partial^2 x} + \alpha u([t],x) + g(t,u([t],x)), & t \in \mathbb{R}_+, x \in [0,\pi], \alpha \in R \\ u(t,0) = u(t,\pi) = 0, & t \in \mathbb{R}_+ \\ u(0) = u_0 \in X. \end{cases}$$

We assume that $(X, ||\cdot||)$=$L^2(0,\pi), \|\cdot\|_2$ and define

$$D(A) = \{\nu, \nu" \in L^2([0,\pi]), \nu(0) = \nu(\pi) = 0\},$$

and

$$A\nu = \nu".$$

A is the infinetesimal generator of a semigroup $T(t)$ on $L^2[0,\phi]$ with

$$\|T(t)\| \leq e^{-t},\ t \geq 0.$$

The operator $A_0 : L^2([0,\pi]) \to L^2([0,\pi])$ defined by $A_0(\nu) = \alpha\nu$ is linear and bounded with $\|A_0\| = |\alpha|$.

Thus, the above partial differential equation can be rewritten as an abstract system of the $Eq.$(1.1), when $u(t)s = u(t,s)$.

THEOREM 5.1. *We assume that $\omega \in \mathbb{N}$, the above system has an unique S-asymptotically ω-periodic mild solution if $|\alpha| + L < 1$.*

Proof. We have $M = 1$, $\delta = 1$, $||A_0|| = |\alpha|$ and apply Theorem 3.5. □

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